Distance between two hyperplanes

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Suppose we have two parallel hyperplanes $\mathcal{L}_1 : \mathbf{w}^{\mathrm{T}}\mathbf{x} + b_1 = 0, \mathcal{L}_2 : \mathbf{w}^{\mathrm{T}}\mathbf{x} + b_2 = 0$, the distance between them is $d = \dfrac{|b_1 - b_2|}{\|\mathbf{w}\|}$

proof: There must exist two points $\mathbf{x}_1, \mathbf{x}_2$, while $\mathbf{x}_1$ is in $\mathcal{L}_1$, and $\mathbf{x}_2$ is in $\mathcal{L}_2$. Also, a pair of $\mathbf{x}_1, \mathbf{x}_2$ satisfies $\|\mathbf{x}_1 - \mathbf{x}_2\| = |b_1 - b_2|$. let $\mathbf{d}$ be a vector perpendicular to $\mathcal{L}_1$ and $\mathcal{L}_2$ and $\|\mathbf{d}\| = d$. Since $\mathbf{w} \perp \mathcal{L}_1$, let $\mathbf{d} = c\mathbf{w}$.

Then we have

$$((\mathbf{x}_2 - \mathbf{x}_1) - c\mathbf{w})^{\mathrm{T}} c\mathbf{w} = 0 \Longrightarrow \mathbf{w}^{\mathrm{T}}(\mathbf{x}_1 - \mathbf{x}_2) - c\|\mathbf{w}\|^2 = 0.$$

Then

$$c = \frac{\mathbf{w}^{\mathrm{T}}(\mathbf{x}_2 - \mathbf{x}_1)}{\|\mathbf{w}\|^2}.$$

Since $\mathbf{w}^{\mathrm{T}}(x_2 - x_1) = b_1 - b_2 \Longrightarrow c = \dfrac{b_1 - b_2}{\|\mathbf{w}\|^2}$. Therefore, $d = \|\mathbf{d}\| = |c|\|\mathbf{w}\| = \dfrac{|b_1 - b_2|}{\|\mathbf{w}\|}$